PARAMETRIC NUMERICAL MOTION STUDY OF THE RIGID BODY WITH A FIXED POINT IN THE EULER-LAGRANGE CASE

STUDIU NUMERIC PARAMETRIC AL MIŞCĂRII RIGIDULUI CU PUNCT FIX ÎN CAZUL LAGRANGE-POISSON

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Abstract: *The Euler-Lagrange case of rigid body with a fixed point has several types of possible motions. These motions are qualitatively studied in many references, but the reader will have to use intuition to understand these motions. Moreover, some characteristics as the nutation amplitude are not determined in existing studies. According to the mechanical properties of the rigid body, in this paper are determined the initial conditions required to obtain each of the known types of motion. Numerical solutions are graphically presented in an easy to understand manner.*
Keywords: *Rigid body, Fixed point, Euler – Lagrange case of integrability*

Rezumat: *Cazul Euler-Lagrange de integrabilitate pentru rigidul cu punct fix conduce la o serie de mişcări posibile. Aceste mişcări sunt studiate din punct de vedere calitativ în multe lucrări, dar cititorul va trebui să se bazeze pe intuiţie pentru a înţelege aceste mişcări. Mai mult, unele caracteristici precum amplitudinea mişcării de nutaţie nu sunt determinate în aceste lucrări. In această lucrare sunt determinate condiţiile iniţiale necesare pentru a se obţine fiecare dintre aceste mişcări în conformitate cu caracteristicile mecanice ale rigidului considerat. Soluţiile numerice sunt determinate grafic într-o manieră uşor de înţeles.*
Cuvinte cheie: *Corp rigid, Punct fix, Cazul de integrabilitate Lagrange – Poisson*

1. INTRODUCTION

The study of the motion of a rigid body having a fixed point dates more than 200 years. However in many recent textbooks, some consequences of these equations are not rigorously deduced and no practical integration results are given. The present study presents the most interesting trajectories of the mass center as functions of the initial conditions.

2. THEORETICAL MODEL

A rigid body of mass m has a fixed point O which is the origin of a fixed Cartesian frame O1x1y1z1, and another Cartesian frame Oxyz attached to the rigid body (Ox, Oy, Oz are the principal axes of inertia) and O1 ≡ O. The Euler angles, θ for nutation, Ψ for precession and φ for rotation (Fig. 1) are used in the following. The projections on the mobile frame of the angular velocities for a body with a fixed point are [1][2]:

$$\begin{aligned} \omega_x &= \dot{\psi}\sin\theta\sin\varphi + \dot{\theta}\cos\varphi \\ \omega_y &= \dot{\psi}\sin\theta\cos\varphi - \dot{\theta}\sin\varphi . \\ \omega_z &= \dot{\varphi} + \dot{\psi}\cos\theta \end{aligned} \tag{1}$$

The equations from the theorem of angular momentum of a rigid body with a fixed point are:

$$\begin{aligned} &J_1\varepsilon_x + \left(J_3 - J_1\right)\omega_y\omega_z = Gh\sin\theta\cos\varphi \\ &J_1\varepsilon_y + \left(J_1 - J_3\right)\omega_z\omega_x = -Gh\sin\theta\sin\varphi, \\ &J_3\varepsilon_z = 0 \Rightarrow \varepsilon_z = 0 \Rightarrow \omega_z = \omega_{z0} = ct \end{aligned} \tag{2}$$

in which G=mg is the weight of the body, h is the distance between the mass centre and the fixed point, J_1 and J_3 are the two distinct central principal mechanical moments of inertia of the body determined about the Oxyz frame.

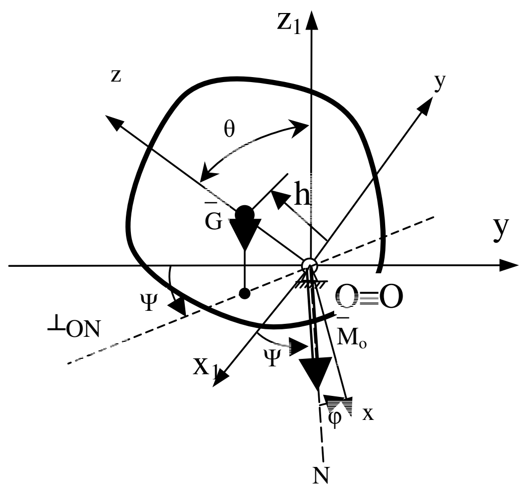

Fig. 1 Rigid body configuration

In the absence of dissipating force, the total energy of rigid body with a fixed point is a constant denoted C_1:

$$\frac{1}{2}J_1\left(\omega_x^2 + \omega_y^2\right) + \frac{1}{2}J_3\omega_{z0}^2 + Gh\cos\theta = C_1 \tag{3}$$

The first two equations from (1) lead to $\omega_x^2 + \omega_y^2 = \dot{\psi}^2\sin^2\theta + \dot{\theta}^2$, which injected in (3) give the expression:

$$\dot{\psi}^2\sin^2\theta + \dot{\theta}^2 = \alpha - \beta\cos\theta, \tag{4}$$

in which α, β are constants, the first one depending on the initial conditions:

$$\alpha = \frac{2C_1 - J_3\omega_{z0}^2}{J_1 + mh^2}; \qquad \beta = \frac{2Gh}{J_1 + mh^2}. \tag{5}$$

From the moment of the weight force, it follows that the projection of the angular momentum of the rigid body with respect to the fixed point O_1 on the O_{z1} axis is a constant. Using the symbol C_2 for this quantity, it follows that:

$$J_1\left(\omega_x\sin\varphi + \omega_y\cos\varphi\right)\sin\theta + J_3\omega_{z0}\cos\theta = C_2. \tag{6}$$

From the first two equations (1) and (6), it can be obtained:

$$\dot{\psi}\sin^2\theta = \gamma - \delta\cos\theta , \tag{7}$$

where:

$$\gamma = \frac{C_2}{J_1}; \qquad \delta = \frac{J_3\omega_{z0}}{J_1} . \tag{8}$$

It is obvious that the equations (4) and (7) lead to:

$$\dot{\theta}^2\sin^2\theta = (\alpha - \beta\cos\theta)\sin^2\theta - (\gamma - \delta\cos\theta)^2 . \tag{9}$$

From the practical point of view of the numerical integration, the time derivative of eq. (9) is necessary, for $\dot{\theta} \neq 0$:

$$\ddot{\theta} = -\dot{\theta}^2 ctg\theta + \frac{1}{\sin\theta}\left[\frac{\beta}{2}\left(1 - 3\cos^2\theta\right) + \left(\alpha + \delta^2\right)\cos\theta - \delta\gamma\right] . \tag{10}$$

The initial conditions can be set in this differential equation as $\theta_{t=0} = \theta_0$; $\dot{\theta}_{t=0} = \dot{\theta}_0$.

3. NUMERICAL SOLUTION

The numerical solutions are obtained by integration using the MATLAB [3] computing program. The classes of solutions are presented in [2], but it is more useful to express these solutions as functions of the initial conditions coupled with the mechanical parameters. The Oz axis is crossing a sphere of radius 1 centered in the fixed point, leaving a trace on the sphere. This trace is an easy way to understand the motion.

If $\alpha - \gamma^2 < 0$ the motion takes place between two parallel circles above the "equator" (Fig. 2), if $\alpha - \gamma^2 > 0$ one tangency circle is below the "equator" (Fig. 3) and if $\alpha - \gamma^2 = 0$ the trajectory is in the "northern hemisphere" and tangent to the "equator".

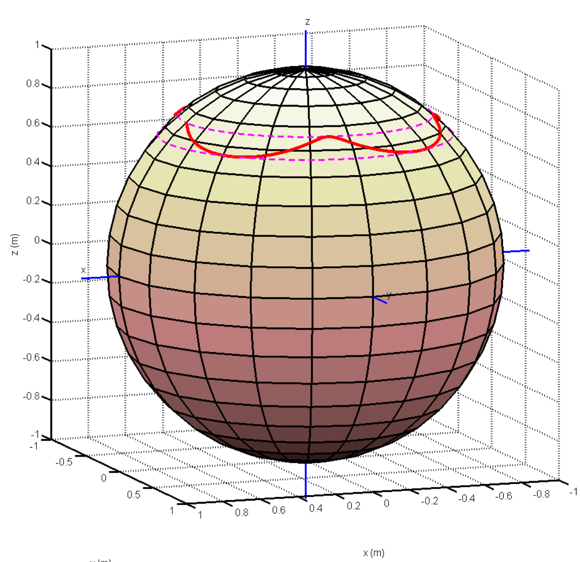

Fig. 2. Wavy motion between two parallel circles

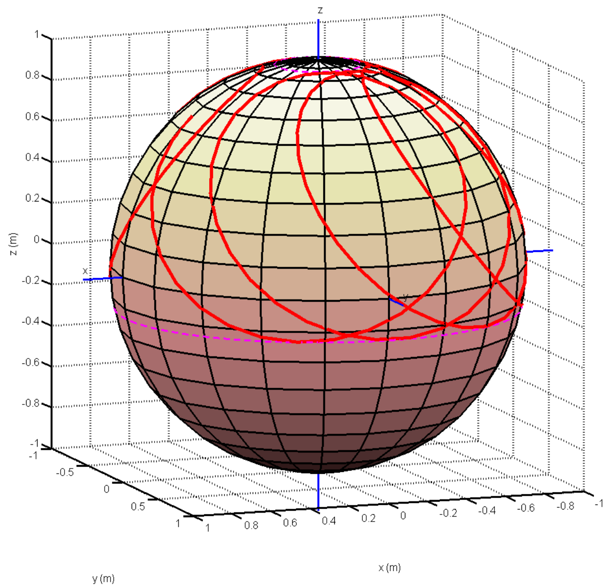

Fig. 3 The path crosses the "equator"

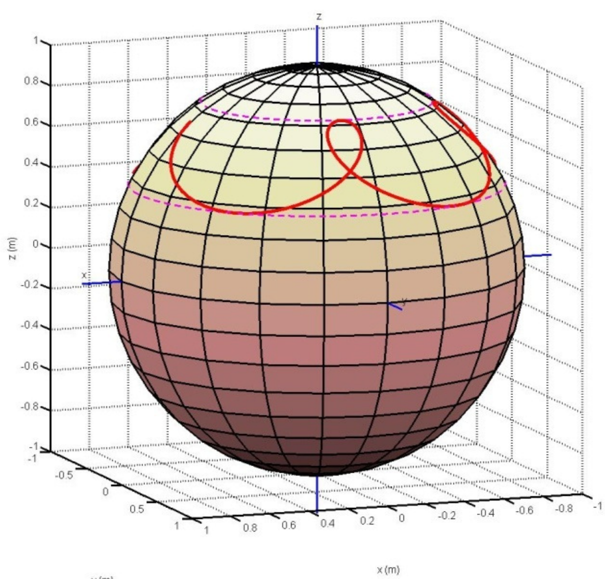

Fig. 4. Motion with loops

If the precession velocity cancels before the nutation velocity, the motion presents loops (Fig. 4). As a particular case, when both velocities cancel at the same moment, there are return points in the path and these points are on the highest "latitude" circle (Fig. 5). Even if the plot cannot present local details, at the return points, the path is tangent to the "meridian". This fact is presented in [1] based on a geometrical interpretation.

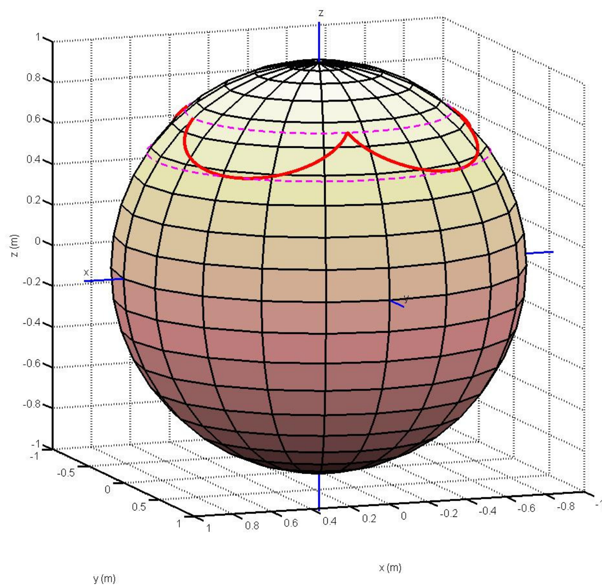

Fig. 5 Motion with return points

One important integration case is the regular precession defined by $\dot{\psi} = const$. After a series of developments, the regular precession can be expressed by the following condition on the initial conditions:

$$0 = 2\left(\alpha + \delta^2\right)\cos\theta_0 - 3\beta\cos^2\theta_0 + \beta - 2\gamma\delta \quad . \tag{11}$$

An example of numerical solution in presented in Fig. 6. The path is a "parallel" circle.

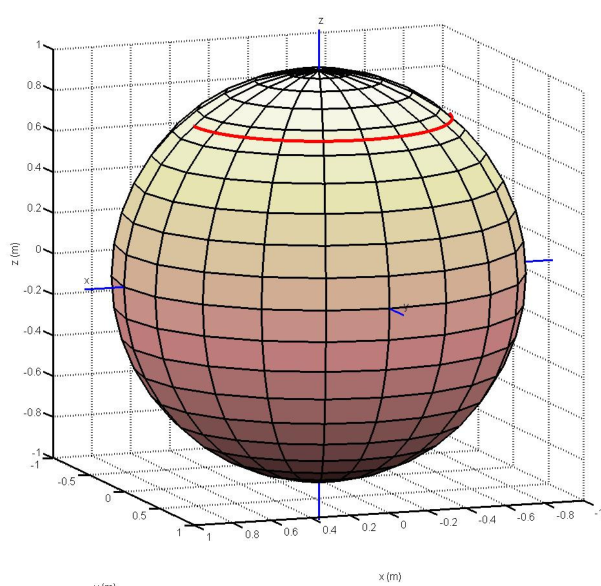

Fig. 6 Regular precession

4. CONCLUSIONS

This study presents the principal motions of a non-centered rigid body with a fixed point in a visually correct manner. The numerical solutions prove their usefulness in understanding the characteristics of the possible motions in this important case of rigid body.

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